Ordinal relative satisficing behavior*

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Abstract: We propose a notion of $r-$rationality, a relative version of satisficing behavior based on the idea that, for any set of available alternatives, individuals choose one of their $r-$best according to a single preference order. We fully characterize the choice functions satisfying the condition for any r, and provide an algorithm to compute the maximal degree of $r-$rationality associated with any given choice function. The notion is extended to individuals whose r may vary with the set of available alternatives. Special cases of ordinal relative satisficing behavior result from a variety of choice models proposed in the literature.

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1 Introduction

In its most classical expression, an individual's choice behavior is said to be rational if it results (1) from choosing the best available alternative according to (2) a complete, reflexive and transitive preference relation on the set of alternatives.

In view of mounting evidence against the observable implications of this simple model of choice, a growing literature has arisen that proposes a variety of departures from it.

We propose a notion of $r-$rationality, based on the idea that the choices of individuals are still guided by a single preference order, but rather than always choosing the very best available alternative, they are content with selecting one of the $r-$best. This proposal provides a purely ordinal and relative version of the classical idea of satisficing behavior. No level of satisfaction is exogenously fixed, agents are not full maximizers, but they follow a clear pattern of behavior whose consequences generate testable implications.

We provide necessary and sufficient conditions for choice functions to be $r-$rationalizable, for any r ranging from 1 to the total number of alternatives.

Let us informally illustrate the basic intuition for our new conditions by first recalling what we know about classical rational choice in our setting, and then comparing its implications with those of the new notion of $r-$rationality. Choice functions satisfying classical rationalizability (now called 1-rationalizability) are usually characterized in our simple setting as the ones satisfying the following necessary and sufficient contraction condition: if an alternative x is chosen for a set B, x must also be chosen from any subset of B that still contains it. This condition provides a "top down" constructive method for the unique rationalization associated to a rationalizable choice function. Just rank in first place the alternative that is chosen when all of them are available, then rank second the one that is chosen after just deleting the first, and so on.

Notice, however, that we could have formulated differently these necessary and sufficient conditions for $1-$rationalizability. Here is another way to describe them, which will in fact be the one inspiring the axioms we use in our general case. Take a choice function. Consider the set of all alternatives that are chosen by that function for some subset of alternatives that is not a singleton. If the choice function is $1-$rationalizable, there must be one and only one alternative that is never chosen, and that should be the one ranking in last place in the rationalizing order. If we look at the family of all non-singleton subsets that do not contain this last alternative, and then at all alternatives that are chosen for some of these subsets, again there must be one and only one alternative that does not appear, and it must be ranked second to last. This "bottom up" construction is the hint to an alternative characterization of classical rationality, through the requirement that choices in a decreasing sequence of sets must exclude one alternative at a time.

Now, the same idea can be tried as a starting point to identify conditions for $r-$rationalizability. For a given choice function, consider all the alternatives that are chosen from subsets containing at least $r+1$ alternatives. If all alternatives were in this set, then the choice function could not be rationalized, because the alternative that is last in an eventual rationalizing order can never be chosen. So, some alternatives must never be chosen, and they must be at most r. If only one is missing, we can assure that it is last in a rationalizing order, if such exists. If several alternatives are never chosen, we will show then if a rational order exists, then there is one that places each one of these unchosen alternatives in the last position. Choose any of them (to be placed last) and look now at all subsets without it that still have cardinality above r. Again, some alternative other than the deleted one must never be chosen out of this restricted family of subsets. Our characterization result is based on a precise formalization of this idea, that clearly extends the axiom that applies to $1-$rationalizable choice functions. Notice, however, that our rationalizations will not be unique unless the choice function is indeed $1-$rationalizable.[1]

Here are two examples that anticipate the kind of issues we deal with.

i) A choice function that is not $1-$rationalizable but is $2-$rationalizable. Let $X = \{a_1, a_2, a_3, a_4\}$. Let F such that for any B with $\{a_1, a_4\} \subseteq B$; $F(B) = a_4$ and otherwise $F(B) = a_i$ where i is the minimum value in $\{1, 2, 3, 4\}$ such that $a_i \in B$. This choice function F is not $1-$rationalizable . We can see this because $F(\{a_1, a_2, a_4\}) = a_4$ and $F(\{a_1, a_2\}) = a_1$ a violation of the standard contraction consistency condition. But we can also see that it cannot be $1-$rationalizable because for any $a_i \in X$, there exists $B \subseteq X$ with $\#B \geq 2$ such that $F(B) = a_i$.

Yet, notice that our choice function F is $2-$rationalizable by the preference orders R, R',where $a_4 R a_2 R a_3 R a_1$ and $a_4 R' a_2 R' a_1 R' a_3$.

ii) A choice function that is neither 1 nor $2-$rationalizable, but is $3-$rationalizable. Let again $X = \{a_1, a_2, a_3, a_4\}$ Let F such that $F(X) = a_3$, $F(B) = a_4$ for any B with $\{a_1, a_4\} \subseteq B \subsetneq X$; and otherwise $F(B) = a_i$ where i is the minimum value in $\{1, 2, 3, 4\}$ such that $a_i \in B$. Notice that this is the same function than in the preceding case, except for its value in X. Yet, now the function is no longer $2-$rationalizable, but is $3-$rationalizable by any order that does not rank a_3 in the last position.

Now, any choice function F on a set of size n is obviously $n-$rationalizable, and in fact also $(n-1)-$rationalizable, as shown later. Hence, we can properly speak about the level of rationality exhibited by any choice function F as given by the minimum value $r(F)$ for

[1] We shall provide a precise statement about non uniqueness in Corollary 1.

which F is $r(F)-$rationalizable. We provide an algorithm to compute the rationality level $r(F)$ associated with any given choice function, thus providing that notion with operational content.

We then propose an even more flexible model of choice, where the level of rationality that an agent displays when choosing from any given set B may vary with the set under consideration. An agent's level of rationality at each set can be described by a function α where $\alpha(B)$ is the rank required for satisfaction when choosing from B. Then, a choice function will be $\alpha-$rationalizable if for some order R, the choice from any set B is among the $\alpha(B)-$best ranked alternatives according to R. And, again, we fully characterize the choice functions that are α rationalizable, for any given α.

The notion that agents may decide to stop short of choosing their best alternative has deep roots and multiple expressions. Recent work on demand theory by Gabaix (2014) and by Aguiar and Serrano (2914) also assumes non-maximizing behavior. In a different vein, Amartya Sen (see Sen (1993), for example) has described the apparently irrational behavior of agents who consistently choose their second best out of the set of alternatives they are proposed. The consequences of that behavior are discussed in Baigent and Gaertner (2010). And our more direct reference point is the idea of satisficing behavior, first introduced by Herbert Simon (1955). When individuals are guided by utility functions, and the comparisons among utility levels have a meaning, one can think of satisficing behavior as the one where the individual chooses among those alternatives that guarantee her a minimum, satisficing level of utility. Our purpose here is to develop a theory of satisficing behavior that is purely ordinal, and thus cannot appeal to any exogenous level of utility as a reference. Within the ordinal context, one could still think of a formulation where some absolute level of satisfaction, identified with the one provided by some exogenously fixed alternative, could set the frontier between satisfactory choices and those that are not. This is the assumption in recent work by Caplin, Dean, and Martin (2011), Papi (2012), Rubinstein and Salant (2006), and Tyson (2008). Our formulation is in a similar spirit, but our notion of a satisficing choice will be relative: agents will select one of the $r-$best ranked alternatives among those available at any act of choice.

Our work, and that of those authors we just mentioned, does not preclude the assumption that agents are still endowed with a preference ordering. Other papers in the burgeoning literature on behavioral economics do, and propose alternative formulations of the actual decision process followed by individuals, as the likely source of their departures from rationality.

Some theories abandon the hypothesis that agents are guided by one order of preferences alone, and consider the possibility that choices might be generated by several preferences, used in some organized manner. These include, for example, Apesteguia and Ballester (2011), De Clippel and Eliaz (2010), Green and Hoiman (2008), Houy and Tadenuma (2009), Kalai, Rubinstein and Spiegler (2002).

Other theories depart from the idea that agents maximize over the set of all feasible alternatives. Observed choices may then be the best elements of some subset of available alternatives, those that have been selected through some screening process. Examples of this approach can be found in Cherepanov, Feddersen and Sandroni(2013), Caplin and Dean (2011), Eliaz, Richter and Rubinstein (2011), Eliaz and Spiegler (2011) Horan (2010), Lleras, Masatlioglu, Nakajima and Ozbay (2010), Manzini and Mariotti (2007 and 2013).

Still another approach is to reformulate the issue of rationality by expanding the set of observables and assuming that information may be available on more complex objects, like sequences of tentative choices over subsets, eventually leading to a final selection. This route is particularly fruitful to model decision processes that involve search costs and stopping rules, and is taken in papers like, Horan (2010), Masatlioglu, Nakajima, and Ozbay (2013), Papi (2012) and Raymond (2013).

Our notions of $r-$ and $\alpha-$rationability can be connected with some behavioral models, and proven to be incompatible with others. For example, when agents can only observe a limited number of the available alternatives, due to search costs or other limitations, they can still guarantee that their choices on a set B will not be ranked below some threshold $\alpha(B)$. Hence, $\alpha-$rationalizability will be among the necessary conditions to be satisfied by choice functions generated by these models. Similarly, processes based on the initial screening of $r-$best elements, followed by a final choice among them, as in Eliaz, Richter and Rubinstein (2011), will generate $r-$rationalizable choice functions. On the other hand, we can prove that our notion of rationality is not implied, nor implies the properties required by other models of choice, like the one proposed by Manzini and Mariotti (2007), for example.[2]

The paper proceeds as follows. After this introduction, in Section 2 we formalize the idea of $r-$rationalizability and provide a first characterization result. In Section 3 we define the degree of rationality of a choice function and provide an algorithm allowing to compute that value. Then, in Section 4 discuss the notion of $\alpha-$rationalizability, to cover the case where the value of r can depend on the set from which the agent can choose. In section 5 we compare our approach with that of two important papers, just to prove by example that our notion of rationalizability cannot be accomodated within some of the alternative

[2] We will elaborate this point in Section 5.

proposals in the behavioral literature, but is compatible with others. Section 6 concludes.

2 $r-$Rationalizable Choice Functions

Consider a finite set X of alternatives with $\#X = n \geq 3$. Let $\mathcal{D} = 2^X - \{\emptyset\}$ be the set of all non-empty subset of alternatives. A choice function on X is a function $F : \mathcal{D} \longrightarrow X$ such that $F(A) \in A$, for every $A \in \mathcal{D}$.

Let R be a preference relation over the set of all alternatives X. Specifically, R is a complete, reflexive, antisymmetric, and transitive binary relation on X.[3]

Given a preference relation R on X and a subset $A \in \mathcal{D}$, let $h(A, R)$ the maximal alternative of a set A with respect to preference R. Formally

$$h(A, R) = x \Leftrightarrow xRy \text{ for every } y \in A.$$

Because R is complete and antisymmetric, $\#h(A, R) = 1$ for every $A \in \mathcal{D}$.

Denote $h^1(A, R) = h(A, R)$, and define for every t;

$$h^t(A, R) = h(A - \bigcup_{i=1}^{t-1} h^i(A, R))$$

and

$$M^r(A, R) = \bigcup_{i=1}^{r} h^i(A, R)$$

$M^r(A, R)$ is the set of elements in A that R ranks in $r-$th position or better.

To relax the assumption that an agent always chooses her best alternative, we say that a choice function is $r-$rationalizable if there exists a preference relation on the alternatives such that the one chosen for each subset is among its $r-$best ranked elements according to that order. Formally,

Definition 1 A choice function F is $r-$*rationalizable* if there exists a preference relation R over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in \bigcup_{i=1}^{r} h^i(A, R) = M^r(A, R).$$

Remark 1 Let r, r' be such that $r \leq r'$ and F a choice function $r-$rationalizable, then F is $r'-$rationalizable.

[3] A binary relation R on F is (i) *complete* if for all $x, y \in X$, either xRy or yRx (ii) *reflexive* if for all $x \in X$; xRx, (iii) *transitive* if for all $x, y, z \in X$ such that $xRyRz$; xRz holds, and (iv) *antisymmetric* if, for all $x, y \in X$ such that xRy and yRx, $x = y$ holds.

Remark 2 The concept of $r-$rationalizability does not impose any restrictions on the possible choices of an agent for sets of size r or less. In particular, any choice function is $n-$rationalizable when there are n alternatives. In fact, any choice function F on a set of size n is $(n-1)-$rationalizable by any preference relation R such that $h(X,R) = F(X)$, since we then have that for any $A \subseteq X$, $F(A) \in M^{(n-1)}(A,R)$. Yet, not every choice function is $(n-2)-$rationalizable, as already proven by the second example in the introduction.

We now introduce definitions leading to our first characterization result, and inspired in the intuitions we provided in the Introduction about our "bottom up" approach.

Given $E \subseteq X$ and r a natural number, define

$$\mathcal{C}_E^r = \{B \in \mathcal{D} : \#(B - E) > r\}$$

Let $E^r = (E_0^r, .., E_{h+1}^r)$ be such that

$$\varnothing = E_0^r \subsetneq E_1^r \subsetneq \cdots \subsetneq E_{h+1}^r = X$$

and

$$\#\left(E_{j+1}^r - E_j^r\right) = \begin{cases} 1 & if \quad 0 \le j < h \\ r & if \quad j = h \end{cases}$$

Let $\mathcal{E}_r$ be the set of all E^r satisficing the above conditions.

Our first theorem provides necessary and sufficient conditions for a choice functions to be $r-$rationalizable.

Theorem 1 Let X be a finite set of alternatives with $\#X = n$ and let r be a natural number with $r \le n$. A choice function F on X is $r-$rationalizable if and only if there exist $E^r \in \mathcal{E}_r$ such that $F(\mathcal{C}_{E_j^r}^r) \subseteq X - E_{j+1}^r$ for any $j = 0, .., h-1$.

Proof $\Rightarrow$) Let F be an $r-$rationalizable choice function. Assume that R^* is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^r(A, R^*).$$

Define $E_1^r = X - M^{n-1}(X, R^*)$ and sequentially

$$E_j^r = X - M^{n-j}(X, R^*)$$

for $j \le n-r$ and $E_{n-r+1}^r = X$. Notice $\varnothing = E_0^r \subsetneq E_1^r \subsetneq \cdots \subsetneq E_{n-r+1}^r = X$, with

$$\#\left(E_{j+1}^r - E_j^r\right) = \begin{cases} 1 & if \quad 0 \le j < n-r \\ r & if \quad j = n-r \end{cases}$$

Let B be such that $B \in \mathcal{C}^r_{E^r_j}$, we must show that $F(B) \in X - E^r_{j+1}$. Because F is an $r-$rationalizable choice function,

$$F(B) \in M^r(B, R^*)$$

Since $\#\left(B - E^r_j\right) > r$, then there exists $z \in \left(B - E^r_j\right)$ such that

$$F(B)R^*z.$$

Because $F(B) \in \left(B - E^r_j\right) \subseteq M^{n-j}(X, R^*)$, we have that $F(B) \in M^{n-j+1}(X, R^*)$, which implies that

$$F(B) \in X - E^r_{j+1}.$$

$\Leftarrow$) Let $\varnothing = E^r_0 \subsetneq E^r_1 \subsetneq \cdot\cdot \subsetneq E^r_{h+1} = X$ be such that

$$\#\left(E^r_{j+1} - E^r_j\right) = \begin{cases} 1 & if \quad 0 \leq j < h \\ r & if \quad j = h \end{cases}$$

and $F(\mathcal{C}^r_{E^r_j}) \subseteq X - E^r_{j+1}$ for any $j = 0, .., h-1$.

Define a preference relation R^* over X as follows::

$$yR^*x \text{ for every } x, y \text{ such that } y \in \left(E^r_j - E^r_{j-1}\right) \text{ and } x \in \left(E^r_i - E^r_{i-1}\right) \text{ with } i < j$$

We will show that F is $r-$rationalizable by R^*. That is, for every $A \in \mathcal{D}$;

$$F(A) \in M^r(A, R^*).$$

Assume otherwise, that there existed $A \in \mathcal{D}$ such that

$$F(A) = z \notin M^r(A, R^*)$$

Notice that xR^*z for every $x \in M^r(A, R^*)$. Thus, there exists $\bar{\imath}$ such that

$$z \in E^r_{\bar{\imath}+1} - E^r_{\bar{\imath}}. \tag{1}$$

Since $M^r(A, R^*) \cup \{z\} \subseteq (A - E^r_{\bar{\imath}})$, then $A \in \mathcal{C}^r_{E^r_{\bar{\imath}}}$. This implies that $F(A) \in X - E^r_{\bar{\imath}+1}$, contradicting (1).

This concludes the proof. ■

Although the formulation of Theorem 1 is in existential terms, there is a very specific constructive element underlying it. The following Remark and the alternative characterization in Theorem 2 are intended to make it more explicit.

Remark 3 Given $Y \subseteq X$, let $\mathcal{T}_Y^r$ be a subset of alternatives that are chosen from any set B such that the cardinality of $(B \cap Y)$ is greater than r. Formally:

$$\mathcal{T}_Y^r = \{x \in Y : F(B) = x, \text{ for some } B \text{ such that } r < \#(B \cap Y)\}. \tag{2}$$

Here is how we can check for rationality and eventually constract the rationalizing order. First look at the set $\mathcal{T}_X^r$ of those alternatives that are chosen for some subset of size at least $(r+1)$. Clearly, there must be at least one that is not: otherwise, F is not $r-$rationalizable, because there is no candidate for last position in a rationalizing order. If that first requirement is satisfied, chose any $x \in X - \mathcal{T}_X^r$ and define $E_1^r = \{x\}$. Let $X_1 = X - E_1^r$. If the size of X_1 is equal to r, define $E_1^r = X$ and we are done F is $r-$rationalizable with elements in X_1 in the top of the rationalizing order. Otherwise, there will be subsets of X_1 with size at least $(r+1)$. Look for those alternatives $\mathcal{T}_{X_1}^r$ that are chosen from some subset B of X with size of $(B \cap X_1)$ larger than r. This set must be strictly smaller than X_1. Otherwise, there is no candidate to be the worse alternative before those in X_1 in the rationalizing order, and F is not $r-$rationalizable. If that second test is still passed, chose any $x \in X_1 - \mathcal{T}_{X_1}^r$ and define $E_2^r = E_1^r \cup \{x\}$. Let $X_2 = X - E_2^r$, if the size of X_2 is equal to r, define $E_3^r = X$ and we are done, F is $r-$rationalizable with elements in X_2 in the top of the rationalizing order. Otherwise, there will be subsets of X_2 with size at least $(r+1)$. Compute the set $\mathcal{T}_{X_2}^r$ of alternatives that now obtain for these subsets. Again, it must be that this new set is smaller than X_2, and so on. Since the necessary conditions in that sequence imply the nestedness of the sets $\mathcal{T}_{X_i}^r$, and X is finite, either they stop holding at some point, with $\#X_i > r$, in which case F is not $r-$rationalizable, or else they lead to a set X_i of size equal to r, and rationalizability holds. Sufficiency is easily derived by ranking different subsets in such a way that those that are still chosen in a certain iteration are ranked above those who have disappeared from the $\mathcal{T}^r$'s in preceding steps.

Our next theorem reformulates our characterization result in terms that are close to the process suggested by Remark 3.

Theorem 2 Let X be a finite set of alternatives with $\#X = n$ and let r be a natural number with $r \leq n$. A choice function F on X is $r-$rationalizable if and only if for any $Y \subseteq X$ with $\#Y > r$, there exists $y \in Y$ such that $F(B) \neq y$ for any $B \subseteq X$ such that $\#(B \cap Y) > r$.

Proof: $\Rightarrow$) Let F be an $r-$rationalizable choice function. Assume that R^* is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^r(A, R^*).$$

Let Y be any subset of alternatives and $\overline{y} \in Y$ such that

$$yR^*\overline{y} \text{ for every } y \in Y.$$

Notice that

$$\overline{y} \notin M^r(B, R^*)$$

for any B such that $\#(B \cap Y) > r$. Thus, $F(B) \neq \overline{y}$.

$\Leftarrow$) Assume that for every $Y \subseteq X$ there exists $y \in Y$ such that $F(B) \neq y$ for any $B \subseteq X$ such that $\#(B \cap Y) > r$.

Consider X. There exists $\overline{y}_1 \in X$ such that $F(B) \neq \overline{y}_1$ for any $B \subseteq X$ such that $\#B > r$.[4] Define

$$E_1^r = \{\overline{y}_1\}.$$

Let X_1 be such that $X_1 = X - E_1^r$. If $\#X_1 = r$, then

$$E_2^r = X.$$

Otherwise, there exists $\overline{y}_2 \in X_1$ such that $F(B) \neq \overline{y}_2$ for any $B \subseteq X$ with $\#(B \cap X_1) > r$. i.e. $\#(B - E_1^r) > r$. Define

$$E_2^r = E_1^r \cup \{\overline{y}_2\}.$$

Notice that $F(B) \notin E_2^r$. Sequentially we can define $E^r = (E_0^r, .., E_{h+1}^r)$ to be such that

$$\varnothing = E_0^r \subsetneq E_1^r \subsetneq \cdot\cdot \subsetneq E_{h+1}^r = X$$

and

$$\#\left(E_{j+1}^r - E_j^r\right) = \begin{cases} 1 & if \quad 0 \leq j < h \\ r & if \quad j = h \end{cases}$$

Let B be such that $B \in \mathcal{C}_{E_j^r}^r$, because $\#\left(B - E_j^r\right) > r$, by construction, $F(B) \notin E_{j+1}^r$. This implies that $F(\mathcal{C}_{E_j^r}^r) \subseteq X - E_{j+1}^r$, for any $j = 0, .., h-1$. By Theorem 1, the choice function F is $r-$rationalizable.

This concludes the proof. ∎

Let us show how we check for rationalizability and at the same time eventually construct a rationalizable order, by examining two examples.

Example 2: Let $X = \{a_1, ..., a_5\}$. The choice function $F : \mathcal{D} \longrightarrow X$ is defined as follows;

[4] In fact, all alternatives in the set $X - \mathcal{T}_X^r$ in Remark 3 will be candidates to play the role of $\overline{y}_1$. A similar degree of freedom will apply when choosing the values $\overline{y}_j$ in the iterative process that is described in what follows.

- If $\#B = 2$, then
  - i) If $B = \{a_1, a_5\}$, then $F(B) = a_1$.
  - ii) Otherwise, let $B = \{a_i, a_j\}$ and $i^* = \min\{i, j\}$; then $F(B) = a_{i^*}$.
- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_4$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$
$F(\{a_1, a_2, a_4, a_5\}) = a_4$	$F(\{a_1, a_3, a_4, a_5\}) = a_4$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_3$.

Let us check whether F is $2-$rationalizable[5].
According to definitions, $E_0^2 = \varnothing$.
Now clearly,

$$X - \mathcal{T}_X^2 = \{x : \nexists B \text{ such that } 2 < \#B \text{ and } F(B) = x\} = \{a_5\}$$

Then $E_1^2 = \{a_5\}$, and $X_1 = X - \{a_5\}$

Next, consider

$$X - \mathcal{T}_{X_1}^2 = \left\{x \in X - E_1^2 : \nexists B \text{ such that } \#\left(B - E_1^2\right) > 2 \text{ and } F(B) = x\right\} = \{a_1, a_2\}.$$

Choose one alternative from the set $\{a_1, a_2\}$, say a_1, and define $E_2^2 = E_1^2 \cup \{a_1\} = \{a_1, a_5\}$ and $X_2 = X - \{a_1, a_5\}$.

Therefore,

$$X - \mathcal{T}_{X_2}^2 = \left\{x \in X - E_1^2 : \nexists B \text{ such that } \#\left(B - E_2^2\right) > 2 \text{ and } F(B) = x\right\} = \{a_2\}.$$

Then define $E_3^2 = E_2^2 \cup \{a_2\} = \{a_1, a_2, a_5\}$.

[5] Notice that F is not $1-$ rationalizable, since for any i there exists B such that $F(B) = a_i$.

Now we can stop, because the size of $X - E_3^2 = X_2$ is equal to 2. Define $E_4^2 = X$ and observe that

$$\varnothing = E_0^2 \subsetneq E_1^2 \subsetneq E_2^2 \subsetneq E_3^2 \subsetneq E_4^2 = X$$

with $\#\left(E_1^2 - E_0^2\right) = \#\left(E_2^2 - E_1^2\right) = \#\left(E_3^2 - E_2^2\right) = 1$ and $\#\left(E_4^2 - E_3^2\right) = 2 = r$, $F(\mathcal{C}^r_{E_j^2}) \subseteq X - E_{j+1}^2$ for any $j = 0, 1, 2, 3$.

Hence F is $2-$rationalizable. □

Example 2 illustrates Theorem 1, with a function F is $2-$rationalizable. Our next example involves a function that is not $2-$rationalizable.

Example 3 Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of alternatives and $F : \mathcal{D} \longrightarrow X$ the choice function defined as follows,

- If $\#B = 2$, then
  - i) If $B = \{a_1, a_5\}$, then $F(B) = a_1$.
  - ii) Otherwise, let $B = \{a_i, a_j\}$ and $i^* = \min\{i, j\}$; then $F(B) = a_{i^*}$.
- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F\left(\{a_1, a_2, a_3, a_4\}\right) = a_1$	$F\left(\{a_1, a_2, a_3, a_5\}\right) = a_3$	$F\left(\{a_1, a_2, a_4, a_5\}\right) = a_4$
$F\left(\{a_1, a_3, a_4, a_5\}\right) = a_3$	$F\left(\{a_2, a_3, a_4, a_5\}\right) = a_3$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_2$.

Let us check whether F is $2-$rationalizable[6].

According to definitions, $E_0^2 = \varnothing$.

Now clearly,

$$X - \mathcal{T}_X^2 = \{x : \nexists B \text{ such that } 2 < \#B \text{ and } F(B) = x\} = \{a_5\}.$$

[6] Notice that F is not $1-$ rationalizable, since for any i there exists B such that $F(B) = a_i$.

Which implies that $E_1^2 = \{a_5\}$ and define $X_1 = X - \{a_5\}$

Notice that

$$X_1 - \mathcal{T}_{X_1}^2 = \left\{x \in X - E_1^2 : \exists B \text{ such that } \#\left(B - E_1^2\right) > 2 \text{ and } F(B) = x\right\} = \{a_1, a_2, a_3, a_4\}.$$

But this means that, for every $y \in Y = \{a_1, a_2, a_3, a_4\}$ we can find B with $\#(B \cap Y) > 2$ such that $F(B) = y$. Theorem 2 implies that F is not $2-$rationalizable. □

Our conditions for $r-$rationalizability allow us to discuss the following issues. What is the number of possible $r-$rationalizations for a given choice function?. How much can we learn about the actual ranking of any given alternative in a given preference order, by observing the choice function of an $r-$rational agent who has that order?. The next Corollary and Remark give answers to these questions.

Using our notation in Remark 3, define sequentially

$$\mathcal{T}_X^r = \{x \in X : F(B) = x, \text{ for some } B \text{ such that } r < \#B\}.$$

$\overline{X}_1 = \mathcal{T}_X^r$, and

$$\mathcal{T}_{\overline{X}_j}^r = \left\{x \in \overline{X}_{j-1} : F(B) = x, \text{ for some } B \text{ such that } \#\left(B \cap \overline{X}_{j-1}\right) > r\right\}.$$

Observe that if a choice function F on X is $r-$rationalizable, then

$$X = X_0 \supsetneq \overline{X}_1 \supsetneq \cdots \supsetneq \overline{X}_{\overline{j}}$$

where $\#\overline{X}_{\overline{j}} = r$.

This allows us to provide an exact count of the number of rationalization that the choice function F will admit.

Corollary 1 Consider an $r-$rationalizable choice function F. This function is rationalizable by exactly $r(F)$ different orders, where

$$r(F) = (r!) \cdot \left[\prod_{j=1}^{\overline{j}} \#\left(\overline{X}_{j-1} - \overline{X}_j\right)\right]$$

The bounds for that number are

$$(r!) \leq r(F) \leq r^q s(r!)$$

where $n = qr+s$, with $0 \leq s < r$ and $q \geq 0$, corresponding to the case where the cardinality of $(X_{j-1} - X_j)$ is minimal and maximal for every j, respectively.

Uniqueness only arises for the classical case where $r = 1$.

Remark 4 Notice that the rank of any alternative in different rationalizations of the same choice function will move between bounds that can be computed from the values of the sets $\overline{X}_j$ in our iterative constructive process. These bounds may be very tight or very loose. For some choice functions the rank of some alternatives in any rationalization may be completely determined, while in some others it may be completely undetermined.

3 The degree of rationality of a choice function

In this section we define a natural measure of the degree of rationality that is exhibited by a choice function F, and we then provide an algorithm that shows how we can actually compute that degree of rationality in an effective manner.

Definition 2 A choice function F exhibits a degree of rationality $r(F)$ iff F is $r(F)-$rationalizable, and it is not $r\acute{}-$rationalizable for any $r\acute{} < r(F)$.[7]

We may naturally associate this degree of rationalizability with the search for a "best approximation" to a fully rational preference, in a similar spirit than Afriat (1973), Houtman and Maks (1985), Varian (1990) or, more recently, Apesteguia and Ballester (2014).

In our case, for any given choice function C, and any given linear order P, compute the vector indicating, for each subset B of alternatives, the rank of $C(B)$ according to P. Find a $\overline{P}$ that minimizes the maximal component of these vectors across all possible preferences, and let $\overline{r}$ be the value of the maximal component of the vector associated to $\overline{P}$. Then, $\overline{r}$ will correspond to C's degree of rationality, and any such $\overline{P}$ is an $\overline{r}-$rationalization for C.

We turn to our proposed algorithm. We do not claim it to be particularly efficient, but it is certainly simple enough to prove that it is possible to associate a degree of rationality to every choice function.

The algorithm follows the basic steps suggested by Remark 3 and Theorem 2. We start by identifying, iteratively, the smallest set size such that, when choosing from all sets of at least that size, the set of alternatives that would be eventually chosen is smaller than the initial set of alternatives. This gives us a first bound on the rationality level. That bound is definitely chosen if no sets of its size or more are left when removing the unchosen alternatives from X.

[7] This notion is in a similar spirit than the exercise in Salant and Rubinstein (2006), where the minimum number of different lists necessary to rationalize a given choice function is also calculated. But of course, we refer to diferent concepts of rationality.

Otherwise, the algorithm continues in a similar manner, but considering only the choices from classes of sets that are nested, and eventually increasing, if necessary, the rationality bound.

Algorithm:
Input: A finite set of alternatives X, with $\#X = n \geq 3$ and F a choice function on X.

Step 0: Set $E_0^r := \varnothing$.

Step 1: For $r = 0, .., n-1$; define

$$\mathcal{T}_X^r = \{x \in X : F(B) = x, \text{ for some } B \text{ such that } r < \#B\}$$

Step 2: Let r_0 be such that $\mathcal{T}_X^{r_0} = X$, $\mathcal{T}_X^{r_0+1} \subsetneq X$. Define $r = r_0 + 1$.
Step 3: Let x be such that $x \in X - \mathcal{T}_X^r$, define $E_1^r = \{x\}$ and $X_1^r = X - E_1^r$ Set $j := 1$.

Step 4: If $\#\left(X - E_j^r\right) = X_j = r$, then define $E_{j+1}^r = X$, $r^* := r$ and go to step 8.

Step 5: Set $j := j + 1$,

$$\mathcal{T}_{X_j}^r = \{x \in X_j : F(B) = x, \text{ for some } B \text{ such that } \#\left(B \cap X_j\right) > r\}$$

Notice that $r < \#X_j$.
Step 6: If $\#\mathcal{T}_{X_j}^r = \#X_j$, set $r := r + 1$ and go to step 3.

Step 7: If $\#\mathcal{T}_{X_j}^r < \#X_{j-1}$, define $E_j^r = E_{j-1}^r \cup \{x\}$ with $x \in \left(X_j - \mathcal{T}_{X_j}^r\right)$. Go to step 4.

Step 8: The choice function F is r^*-rationalizable. Define $r^* = r(F)$.

END.

Theorem 3 The natural number $r(F)$ is the minimum such that the function F is $r(F)-$ rationalizable.
Proof Let $r^* = r(F)$ be obtained in step 7.

First, we will prove that F is r^*-rationalizable. It follows clearly from step 4 and Theorem 1 because there exists $E \in \mathcal{E}_{r^*}$ such that $F(\mathcal{C}_{E_j^{r^*}}^r) \subseteq X - E_{j+1}^{r^*}$ for any $j = 0, .., h-1$.

Now, we have to prove that F is not $(r^* - 1)-$rationalizable. Assume otherwise, that there exists R such that for any $Y \subset X$;

$$F(Y) \in M^{(r^*-1)}(Y, R). \tag{3}$$

The algorithm stops for $r = r^*$. Then for $r = r^* - 1$ the algorithm did not lead to Step 8, but proceeded to steps 5 and 6. But then, the only chance for the algorithm to finally stop after having re-visited step 4 is that at some point it reverted to step 3, and this implies that there exists j such that $r < \#\mathcal{T}^r_{X_j}$ and $\mathcal{T}^r_{X_j} = X_j$.Notice that, for $Y = X_j$ $(\#Y > r)$, and for any $x \in Y$ $\exists B$ such that $\#(B \cap Y) > r$, and $F(B) = x$. But then, Theorem 2 implies that F is not $(r^* - 1)-$rationalizable.

This concludes the proof. ■

We illustrate how the algorithm works with the following example.

Example 4 Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of alternatives and $F : \mathcal{D} \longrightarrow X$ the choice function defined as follows for each subset of size 2 or larger:

- $F(\{a_i, a_j\}) = a_j$, with $j > i$
- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_1$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_1, a_2, a_4, a_5\}) = a_4$
$F(\{a_1, a_3, a_4, a_5\}) = a_3$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_2$.

The algorithm

Step 0: Set $E_0 = \varnothing$ and $X = \mathcal{T}_0$.

Step 1: For $r = 0, .., 4$; define

$$\mathcal{T}^r_X = \{x \in X : F(B) = x, \text{ for some } B \text{ such that } r < \#B\}$$

That is: $\mathcal{T}^0_X = \mathcal{T}^1_X = X$; $\mathcal{T}^2_X = \{a_1, a_2, a_3, a_4\}$; $\mathcal{T}^3_X = \{a_1, a_2, a_3, a_4\}$; $\mathcal{T}^4_X = \{a_2\}$.

Step 2: Set $\mathcal{T}^1_{X_1} = \mathcal{T}^0_X = X$ and $\mathcal{T}^2_{X_1} \subsetneq X$, $r_0 = 1$ and $r = 2$.

Step 3: Define $E^2_1 = \{a_5\}$ and $X_1 = X - \{a_5\}$.

Step 4: Because $\#(X_1) > 2$, then go to step 5.

Step 5: Define

$$\mathcal{T}^2_{X_1} = \{x \in X_1 : F(B) = x, \text{ for some } B \text{ such that } \#(B \cap X_1) > 2\} = \{a_1, a_2, a_3, a_4\}.$$

Step 6: Because $\mathcal{T}^2_{X_1} = X_1 = (X - E^2_1) = \{a_1, a_2, a_3, a_4\}$, then define $r = 3$ and go to step 3. Notice F is not $2-$rationalizable.

Step 3: Because $\mathcal{T}^3_X = \{a_1, a_2, a_3, a_4\}$, define $E^3_1 = \{a_5\}$ and $X_1 = X - \{a_5\}$ then go to step 4.

Step 4: Because $\#(X - E^3_1) > 3$, then go to step 5.

Step 5: Define

$$\mathcal{T}^3_{X_1} = \{x \in X_1 : F(B) = x, \text{ for some } B \text{ such that } \#(B \cap X_1) > 3\} = \{a_1, a_2\}.$$

Because $\#\mathcal{T}^3_{X_1} < \#X_1$, go to step 7.

Step 7: Because $a_4 \in \left(X_1 - \mathcal{T}^3_{X_1}\right)$. Define $E^3_2 = E^3_1 \cup \{a_4\} = \{a_4, a_5\}$ and $X_2 = X - \{a_4, a_5\}$. Go to step 4.

Step 4: Because $\#X_2 = 3 = r$, then define $E^3_3 = X_2$ and go to step 8.

Step 8: The choice function F is $3-$rationalizable. Notice that

$$\varnothing = E^3_0 \subsetneq E^3_1 = \{a_5\} \subsetneq E^3_2 = \{a_4, a_5\} \subsetneq E^3_3 = X.$$

END.

4 A further extension of the rationalizability concept

In this section we consider the general case where the same agent may be content, or not, with getting her $r-$ranked alternative, depending on the context where this choice occurs. For example, a larger r may be required when choosing from a set of similar alternatives, while a smaller level of r may apply when the states involved when making a potential mistake are larger. Our definitions and results are similar to those already presented, and we shall thus be a bit more expedient in the presentation. The proofs are relegated to the appendix, but examples are provided to illustrate the main aspects of the proposed extension.

Consider a finite set X of alternatives with $\#X \geq 3$, and a function $\alpha : \mathcal{D} \longrightarrow \{1,..,n\}$ that determines a level of relative ordinal satisficing behavior for each subset B of alternatives. We say that a choice function is $\alpha-$rationalizable if there exists a preference relation on the alternatives such that the one chosen for each subset A of alternatives is among the $\alpha(A)-$first ranked elements of the order among those in the sets. Formally:

Definition 3 A choice function F is *α-rationalizable* if there exists a preference relation R over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in \bigcup_{i=1}^{\alpha(A)} h^i(A,R) = M^{\alpha(A)}(A,R).$$

Given a natural number $r \in \mathbf{N}$, we say that F is $r-$rationalizable if it is $\alpha-$rationalizable with $\alpha(B) = r$.

Given $E \subseteq X$ and a function $\alpha : \mathcal{D} \longrightarrow \{1,..,n\}$, define

$$\mathcal{C}^{\alpha}_{E} = \{B \in 2^X : \#(B-E) > \alpha(B)\}$$

Let $E^{\alpha} = (E^{\alpha}_0, .., E^{\alpha}_{h+1})$ be such that

$$\varnothing = E^{\alpha}_0 \subsetneq E^{\alpha}_1 \subsetneq \cdot\cdot \subsetneq E^{\alpha}_{h+1} = X$$

and

$$\#\left(E^{\alpha}_{j+1} - E^{\alpha}_{j}\right) = \begin{cases} 1 & if \quad 0 \leq j < h \\ r & if \quad j = h \end{cases}$$

r have to be such that $\mathcal{C}^{\alpha}_{E^{\alpha}_h} = \varnothing$.[8] Define $\mathcal{E}_{\alpha}$ the set of all E^{α} satisficing the above conditions.

Our theorems provide necessary and sufficient conditions for a choice function to be α-rationalizable.

Theorem 4 Let X be a finite set of alternatives with $\#X = n$ and a function α. A choice function F on X is $\alpha-$rationalizable if and only if there exist $E^{\alpha} \in \mathcal{E}_{\alpha}$ such that $F(\mathcal{C}^{\alpha}_{E^{\alpha}_j}) \cap E^{\alpha}_{j+1} = \varnothing$ for any $j = 0,..,h-1$.

Proof See appendix.

Again, we can reformulate the characterization in a form that is parallel to Theorem 2. ■

Theorem 5 Let X be a finite set of alternatives with $\#X = n$ and a function α. A choice function F on X is $\alpha-$rationalizable if and only if for every $Y \subseteq X$ there exists $y \in Y$ such that $F(B) \neq y$ for any $B \subseteq X$ such that $\#(B \cap Y) > \alpha(B)$.

Proof See appendix. ■

[8]Observe that $r \geq \min_{\{B \subseteq X : \alpha(B) < \#B\}} \alpha(B)$.

Notice that given a function $\alpha(B) = \#B$, every choice function F is $\alpha-$rationalizable. Moreover, let α, α' be such that $\alpha(B) \leq \alpha'(B)$ for every B. If the choice function F is $\alpha-$rationalizable, then F is $\alpha'-$rationalizable.

5 Two comparative examples

5.1 A comparison of $r-$rationalizability with the rationale of shortlist methods

As an illustration that our notion of rationalizability characterizes behavior that is independent from the one predicated by other models, we compare its implications with those of the following celebrated proposal by Manzini and Mariotti (2006).

Definition 4 A choice function F is a Rational Shortlist Method (RSM) whenever there exists an ordered pair (P_1, P_2) of asymmetric relations, with $P_i \subseteq X \times X$ such that

$$F(A) = \max(\max(A; P_1); P_2).$$

Manzini and Mariotti show that choice functions of this form do not need to be $1-$rational. Yet, they identify two properties that fully characterize them.

Expansion: For all $S, T \subseteq X$, if $x = F(S) = F(T)$, then $x = F(S \cup T)$.

WARP* : If $\{x, y\} \subseteq R \subseteq S$ and $F(S) = F(\{x, t\}) = x$, then $y \neq F(R)$.

Theorem (Manzini-Mariotti): The choice function F is RSM if and only if it satisfies WARP* and Expansion.

We'll show that their choice functions may not satisfy our notion of rationality and, conversely, that our functions need not be of their form.

Example 5 The example shows a 2-rationalizable choice function F that is not RSM. Let $X = \{x_1, x_2, x_3, x_4\}$, P an strict order on X; $x_1 P x_2 P x_3 P x_4$, and F be a choice function defined as follows:

$$F(A) = \begin{cases} \max(A; P) & if \quad A \neq \{x_1, x_2, x_3\} \\ x_2 & if \quad A = \{x_1, x_2, x_3\} \end{cases}$$

Observe that F is $2-$rationalizable but is not RSM. This is because $F(\{x_1, x_2, x_3\}) = x_2$, $F(\{x_2, x_3, x_4\}) = x_2$, $F(X) = x_1$, and therefore F does not satisfy expansion.

Example 6 The following example shows that there exists a RSM choice function F, that is not 2-rationalizable. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of alternatives. Let R_1 be the following partial order;

$$x_4 R_1 x_1 \text{ and } x_5 R_1 x_2.$$

Let R_2 be a strict order on X;

$$x_1 R_2 x_2 R_2 x_3 R_2 x_4 R_2 x_5$$

Observe that,

$$\max(\{x_1, x_2, x_3\}; R_1) = \{x_1, x_2, x_3\},$$

$$\max(\{x_1, x_2, x_3, x_4\}; R_1) = \{x_2, x_3, x_4\},$$

and

$$\max(\{x_1, x_2, x_3, x_4, x_5\}; R_1) = \{x_3, x_4, x_5\}.$$

Observe that for every $S \subseteq X$;

$$F(S) = \max\left(\max(S; R_1), R_2\right).$$

Then,

$$F\left(\{x_1, x_2, x_3\}\right) = x_1,$$

$$F\left(\{x_1, x_2, x_3, x_4\}\right) = x_2,$$

and

$$F\left(\{x_1, x_2, x_3, x_4, x_5\}\right) = x_3.$$

Let $Y = \{x_1, x_2, x_3\}$, observe that for each $x_i \in Y$ there B such that $\#(B \cap Y) \geq 3$ and $F(B) = x_i$. Theorem 2, implies that F is not $2-$rationalizable.

The above examples show that the two models lead to independent restrictions on the set of possible choice functions they generate.

5.2 Choosing from a set of chosen finalists

We can connect our work with that of Eliaz, Richter and Rubinstein in their article "Choosing the two finalists" (2011). These authors characterize "top two" correspondences that select the best two outcomes from an order, given each subset of alternatives. We can extend our proposed notion of rationalizability and say that these correspondences are $2-$rationalizable. And then, one can see that our set of $2-$rationalizable choice functions would consist of all selections from some $2-$rationalizable choice correspondence. In a natural manner, $r-$rationalizable choice correspondences could also be defined and characterized, and selections from them would coincide with our $r-$rationalizable choice functions.

Hence, our notion of $r-$rationalizability is a necessary conditions for all those processes that may be described as choosing r finalists in a first stage, and then using some additional criterion to narrow down the choice from any set to a singleton. Of course, additional restriction could be imposed on choice functions when being specific about the criterion for final selection. Just to illustrate this point, here are some suggestions on how to complement the choice of two finalists.

Assume, for example, that the final choice between two finalists was to be made by a committee that uses majority rule with tie breaking. Given any preference profile on the set of alternatives X, the committee's majority rule will be a tournament, and for each pair of finalists the tournament will be used as the selection device. Then, clearly the choice function that we shall obtain will be $2-$rationalizable, but not any $2-$rationalizable choice function could be derived from that process. This is because the committee will choose the same alternative out of each pair x, y regardless of the set B from which these two were picked as finalists. Thus, a choice function selected by majority from a two-top correspondence will satisfy the following additional condition.

Condition 1 Given $\{a, b\} \subsetneq B$ and $F(\{a, b\}) = a \neq F(B) = b$, there must exist $c \in B - \{a, b\}$ such that $a \neq F(C)$, for every C with $\{a, b, c\} \subseteq C$.

In fact, rationalizability plus condition 1 fully characterize the set of choice functions that can be obtained from the two-stage process we describe, provided the number of committee members is large enough. We express this fact as follows.

Remark 6 A choice function F can be generated by choosing the best two candidates from an order and then selecting the one that has a majority in a committee of size larger than the total number of alternatives if and only if it is $2-$rationalizable and satisfies condition 1.

We leave the detailed proof of the Remark 6 to the reader, but note that it relies on the fact that any given tournament on a set of alternatives X can be generated as the majority relation for some committee whose size is larger than that of X (McGarvey (1953), Stearns (1959), see also Moulin (1988)).

A further narrowing down of the preceding class would obtain if we required the selection to be made by choosing the maximal element of a transitive relation, not necessarily the same as the one used to choose the finalists. This particular case, where further restrictions should be imposed on the resulting choice function, has been studied by Bajraj and Ülkü (2014).

Our main point is made through this analysis of the case for two finalists. These screening processes, coupled with a criterion to choose from pairs, generate $2-$rationalizable choice functions, which may be more or less restricted in scope depending on the second stage selection criterion.

Similarly, the choice of $r-$finalists, along with a choice function on subsets of size r, would give rise to $r-$rationalizable choice functions, whose additional properties will be determined by the choice function in question.

6 Conclusions

We conclude by acknowledging some of the limitations of our present approach and by suggesting some further lines of work.

Our analysis is limited to finite sets of alternatives. Extending the notion of second best or of $r-$best alternatives to sets with a continuum of alternatives is non-trivial. We have already mentioned the important and recent literature on demand theory that also considers non-maximizing agents (Gabaix X. 2014, Aguiar V. and Serrano R., 2014). Obviously, it starts from an opposite end, where a continuous set of alternatives is the natural assumption. Even if these two ends do not meet, we feel that our very simple formulation of the basic choice problem is also a natural starting point. In particular, Aguiar and Serrano's definition of the "size" of bounded rationality is in a similar spirit than our calculations of the rationality level of a choice function, in Section 3.

We also limit attention to choice functions, and one may want to see similar results for the case of correspondences. For example, in the case of $2-$rationalizability, we may want to characterize the behavior of agents may who choose several alternatives belonging to their best and second best indifference classes within a set. That would be consistent with the assumption that agents' preferences may be weak orders. There are several ways to make this idea more precise, and we have partial results in the same spirit as the ones presented here.

On the positive side, the assumption that we have information on the choices over all possible sets is not a limitative one. We can still discuss the rationalizability of choice functions defined on any family of subsets, by just assuming that our α function assigns to those sets on which we have no information a value equal to its cardinality.

Finally, if one was convinced that the present proposal is a reasonable alternative to full rationality, it would be worth investigating the consequences on the theory of games that would derive from assuming that agents behave accordingly.

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7 Appendix

Theorem 4 Let X be a finite set of alternatives with $\#X = n$ and a function α. A choice function F on X is $\alpha-$rationalizable if and only if there exists $E \in \mathcal{E}_\alpha$ such that $F(\mathcal{C}^{\alpha}_{E^{\alpha}_j}) \cap E^{\alpha}_{j+1} = \varnothing$ for any $j = 0, .., h-1$.
Proof $\Rightarrow$) Let F be an $\alpha-$rationalizable choice function. Assume that R^* is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^{\alpha(A)}(A, R^*).$$

Define $E^{\alpha}_1 = X - M^{n-1}(X, R^*)$ and sequentially

$$E^{\alpha}_j = X - M^{n-j}(X, R^*)$$

for $j \leq n-r$ and $E^{\alpha}_{n-r+1} = X$, where $r = \min_{\{B \subseteq X : \alpha(B) < \#B} \alpha(B)$. Notice that $\varnothing = E^{\alpha}_0 \subsetneqq E^{\alpha}_1 \subsetneqq \cdot\cdot \subsetneqq E^{\alpha}_{n-r+1} = X$, with

$$\#\left(E^{\alpha}_{j+1} - E^{\alpha}_j\right) = \begin{cases} 1 & if \quad 0 \leq j < n-r \\ r & if \quad j = n-r \end{cases}$$

Let B be such that $B \in \mathcal{C}^{\alpha}_{E^{r}_j}$, we have to show that $F(B) \notin E^{r}_{j+1}$. Because F is an $\alpha-$rationalizable choice function,

$$F(B) \in M^{\alpha(B)}(B, R^*)$$

Since $\#\left(B - E^{r}_j\right) > \alpha(B)$, there exists $z \in \left(B - E^{\alpha}_j\right)$ such that

$$F(B) R^* z.$$

Because $z, F(B) \in \left(B - E^{\alpha}_j\right) \subseteq M^{n-j}(X, R^*)$, we have that $F(B) \in M^{n-j+1}(X, R^*)$ which implies that

$$F(B) \notin E^{\alpha}_{j+1}.$$

$\Leftarrow$) Let $\varnothing = E_0^\alpha \subsetneq E_1^\alpha \subsetneq \cdot\cdot \subsetneq E_{h+1}^\alpha = X$ be such that

$$\#\left(E_{j+1}^\alpha - E_j^\alpha\right) = \begin{cases} 1 & if \quad 0 \leq j < h \\ r & if \quad j = h \end{cases}$$

and $F(\mathcal{C}_{E_j^\alpha}^\alpha) \cap E_{j+1}^\alpha = \varnothing$ for any $j = 0, .., h-1$.

Define a preference relation R^* over X as follows:

yR^*x for every x, y such that $y \in \left(E_j^\alpha - E_{j-1}^\alpha\right)$ and $x \in \left(E_i^\alpha - E_{i-1}^\alpha\right)$ with $i < j$

We will show that F is $\alpha-$rationalizable by R^*. That is, for every $A \in \mathcal{D}$;

$$F(A) \in M^{\alpha(A)}(A, R^*).$$

Assume otherwise, that there existed $A \in \mathcal{D}$ such that

$$F(A) = z \notin M^{\alpha(A)}(A, R^*)$$

Notice that xR^*z for every $x \in M^{\alpha(A)}(A, R^*)$. Thus, there exists $\bar{i}$ such that

$$z \in E_{\bar{i}+1}^\alpha - E_{\bar{i}}^\alpha. \tag{4}$$

Since $M^{\alpha(A)}(A, R^*) \cup \{z\} \subseteq \left(A - E_{\bar{i}}^\alpha\right)$, then $A \in \mathcal{C}_{E_{\bar{i}}^\alpha}^\alpha$. This implies that $F(A) \notin E_{\bar{i}+1}^r$, contradicting (4).

This concludes the proof. ■

Theorem 5 Let X be a finite set of alternatives with $\#X = n$ and a function α. A choice function F on X is $\alpha-$rationalizable if and only if for every $Y \subseteq X$ there exists $y \in Y$ such that $F(B) \neq y$ for any $B \subseteq X$ such that $\#(B \cap Y) > \alpha(B)$.

Proof: $\Rightarrow$) Let F be an $\alpha-$rationalizable choice function. Assume that R^* is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^{\alpha(A)}(A, R^*).$$

Let Y be any subset of alternatives and $\overline{y} \in Y$ such that

$$yR^*\overline{y} \text{ for every } y \in Y.$$

Notice that for every B such that $\#(B \cap Y) > \alpha(B)$, we have that

$$\overline{y} \notin M^{\alpha(B)}(B, R^*).$$

Thus, $F(B) \neq \overline{y}$.

$\Leftarrow$) Assume that for every $Y \subseteq X$ there exists $y \in Y$ such that $F(B) \neq y$ for any $B \subseteq X$ such that $\#(B \cap Y) > \alpha(B)$.

Consider X. Then there exists $\overline{y}_1 \in X$ such that $F(B) \neq \overline{y}$ for any $B \subseteq X$ such that $\#B > \alpha(B)$. Define

$$E_1^{\alpha} = \{\overline{y}_1\}.$$

Consider $X_1 = X - E_1^{\alpha}$. If for any B, $\#(X_1 \cap B) < \alpha(B)$, then define

$$E_2^{\alpha} = X.$$

Otherwise, there exist $\overline{y}_2 \in X_1$ such that $F(B) \neq \overline{y}_2$ for any $B \subseteq X$ with $\#(B \cap X_1) > \alpha(B)$. i.e. $\#(B - E_1^{\alpha}) > \alpha(B)$. Define

$$E_2^{\alpha} = E_1^{\alpha} \cup \{\overline{y}_2\}.$$

Notice that $F(B) \notin E_2^{\alpha}$. Sequentially we can define, $E^{\alpha} = (E_0^{\alpha}, .., E_{h+1}^{\alpha})$ be such that

$$\varnothing = E_0^{\alpha} \subsetneq E_1^{\alpha} \subsetneq \cdot\cdot \subsetneq E_{h+1}^{\alpha} = X$$

and

$$\#\left(E_{j+1}^{\alpha} - E_j^{\alpha}\right) = \begin{cases} 1 & if \quad 0 \leq j < h \\ r & if \quad j = h \end{cases}$$

Let B be such that $B \in \mathcal{C}_{E_j^{\alpha}}^{\alpha}$. Since $\#\left(B - E_j^{\alpha}\right) > \alpha(B)$, by construction, $F(B) \notin E_{j+1}^{\alpha}$. This implies that $F(\mathcal{C}_{E_j^{r}}^{\alpha}) \cap E_{j+1}^{\alpha} = \varnothing$, for any $j = 0, .., h-1$. Theorem 4, implies that the choice function F is $\alpha-$rationalizable.

This concludes the proof. ■

The following example illustrates the family of subset of alternatives that we are constructing.

Example 7: $X = \{a_1, ..., a_5\}$ the set of alternatives. The choice function $F : \mathcal{D} \longrightarrow X$ is defined as follows; for each subset of size 3 or larger:

- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_4$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_1, a_2, a_4, a_5\}) = a_4$
$F(\{a_1, a_3, a_4, a_5\}) = a_1$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_3$.

Let $\alpha : \{1, .., 5\} \to \{1, .., 5\}$ be defined by

$\alpha(1) = 1$	$\alpha(2) = 2$	$\alpha(3) = 2$	$\alpha(4) = 3$	$\alpha(5) = 1$

That is, it has the same value for two subsets of equal cardinality.

Now clearly,

$$\mathcal{T}_X^\alpha = \{x \in X : F(B) = x, \text{ for some } B \text{ such that } \alpha(\#B) < \#B\} = \{a_1, a_2, a_3, a_4\}$$

This implies that $E_1^\alpha = \{a_5\}$. Define $X_1 = X - \{a_5\} = \{a_1, a_2, a_3, a_4\}$.

Notice that

$$\mathcal{T}_{X_1}^\alpha = \{x \in X_1 : F(B) = x, \text{ for some } B \text{ such that } \#(B \cap X_1) > \alpha(\#B)\} = \{a_3, a_4\}.$$

Because $a_1 \in X_1 - \mathcal{T}_{X_1}^\alpha$, define $E_2^\alpha = E_1^\alpha \cup \{a_5\} = \{a_1, a_5\}$ and $X_2 = (X - E_2^\alpha) = \{a_2, a_3, a_4\}$.

Consider

$$\mathcal{T}_{X_2}^\alpha = \{x \in X_2 : F(B) = x, \text{ for some } B \text{ such that } \#(B \cap X_2) > \alpha(\#B)\} = \{a_3\}.$$

Since $a_2 \in X_2 - \mathcal{T}_{X_2}^\alpha$, define $E_3^\alpha = E_2^\alpha \cup \{a_2\} = \{a_1, a_2, a_5\}$ and $X_3 = X - \{a_1, a_2, a_5\}$

Therefore,

$$\mathcal{T}_{X_3}^\alpha = \{x \in X_3 : F(B) = x, \text{ for some } B \text{ such that } \#(B \cap X_3) > \alpha(\#B)\} = \{a_3\}.$$

Since $a_4 \in X_3 - \mathcal{T}_{X_3}^\alpha$, then set $E_4^\alpha = E_3^\alpha \cup \{a_4\} = \{a_1, a_2, a_4, a_5\}$ and $X_4 = X - \{a_1, a_2, a_4, a_5\}$. Clearly $E_5^\alpha = X$.

Therefore $\varnothing = E_0^\alpha \subsetneq E_1^\alpha \subsetneq E_2^\alpha \subsetneq E_3^\alpha \subsetneq E_4^\alpha \subsetneq E_5^\alpha = X$, and by theorem 5 the function F is $\alpha-$rationalizable. $\square$